MATH 2028 Honours Advanced Calculus II
2024-25 Term 1
Problem Set 4

due on Nov 1, 2024 (Friday) at 11:59PM

Instructions: You are allowed to discuss with your classmates or seek help from the TAs but you are required to write/type up your own solutions. You can either type up your assignment or scan a copy of your written assignment into ONE PDF file and submit through Blackboard on/before the due date. Please remember to write down your name and student ID. **No late homework will be accepted.**

Problems to hand in

1. Calculate the line integral $\int_C f ds$ and $\int_C \vec{F} \cdot d\vec{r}$ where
   (a) $f(x,y,z) = y^2 + z - 3xy$, $\vec{F}(x,y,z) = (y^2, z, -3xy)$ and C is the line segment from $(1,0,1)$ to $(2,3,-1)$.
   (b) $f(x,y) = x + y$, $\vec{F}(x,y) = (-y^3, x^3)$ and C is the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ oriented counterclockwise.
2. Let C be the curve of intersection of the upper hemisphere $x^2+y^2+z^2 = 4$, $z \geq 0$ and the cylinder $x^2 + y^2 = 2x$, oriented counterclockwise as viewed from high above the xy-plane. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = (y,z,x)$.
3. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ is the vector field
$$\vec{F}(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$
   and C is an arbitrary path from $(1,1)$ to $(2,2)$ not passing through the origin.
4. Determine which of the following vector field $\vec{F}$ is conservative on $\mathbb{R}^n$. For those that are conservative, find a potential function f for it. For those that are not conservative, find a closed curve such that $\oint_C \vec{F} \cdot d\vec{r} \neq 0$.
   (a) $\vec{F}(x,y) = (y^2, x^2)$;
   (b) $\vec{F}(x,y,z) = (y^2 z, 2xyz + \sin z, xy^2 + y\cos z)$.
5. Find the area of the region enclosed by the curve $x^{2/3} + y^{2/3} = 1$.

Suggested Exercises

1. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where
   (a) $\vec{F}(x,y,z) = (z,x,y)$ and C is the line segment from $(0,1,2)$ to $(1,-1,3)$.
   (b) $\vec{F}(x,y,z) = (y,0,0)$ where C is the intersection of the unit sphere $x^2+y^2+z^2 = 1$ and the plane $x+y+z = 0$, oriented counterclockwise as viewed from high above the xy-plane.

2. Calculate $\int_C F \cdot d\vec{r}$ where $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector field

$$\vec{F}(x, y, z) = \left(3x + y^2 + 2xz, 2xy + ze^{yz} + y, x^2 + ye^{yz} + ze^{z^2}\right)$$

and C is the parametrized curve $\gamma : [0, 1] \to \mathbb{R}^3$ given by

$$\gamma(t) = \left(e^{t^7 \cos(2\pi t^{21})}, t^{17} + 4t^3 - 1, t^4 + (t - t^2)e^{\sin t}\right).$$

3. Compute the line integral $\int_C F \cdot d\vec{r}$ where
   (a) $\vec{F}(x, y) = (xy^3, 0)$ and C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise;
   (b) $\vec{F}(x, y) = (-y\sqrt{x^2 + y^2}, x\sqrt{x^2 + y^2})$ and C is the circle $x^2 + y^2 = 2x$ oriented counterclockwise.

4. Let C be the circle $x^2 + y^2 = 2x$ oriented counterclosewise. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where

$$\vec{F}(x, y) = \left(-y^2 + e^{x^2}, x + \sin(y^3)\right).$$

5. Find the area of the region enclosed by the curve

$$\gamma(t) = \big(\cos t + t \sin t, \sin t - t \cos t\big), \quad 0 \le t \le 2\pi$$

   and the line segment from $(1, -2\pi)$ to $(1, 0)$.

6. Let $0 < b < a$. Find the area under the curve $f(t) = (at - b \sin t, a - b \cos t)$, $0 \le t \le 2\pi$, above the x-axis.

7. Suppose C is a piecewise C^1 closed curve in $\mathbb{R}^2$ that intersects with itself finitely many times and does not pass through the origin. Show that the line integral

$$\frac{1}{2\pi} \int_C -\frac{y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy$$

   is always an integer. This is called the *winding number* of C around the origin.

Challenging Exercises

1. Suppose $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field on $\mathbb{R}^n$ defined by

$$\vec{F}(x_1, x_2, \cdots, x_n) = (f(r)x_1, f(r)x_2, \cdots, f(r)x_n)$$

   where $f : \mathbb{R} \to \mathbb{R}$ is a given function and $r := \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$.
   (a) Suppose f is differentiable everywhere. Prove that for all $i, j = 1, \cdots, n$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$$

   on $\mathbb{R}^n \setminus \{\vec{0}\}$ where F_k is the k-th component function of the vector field F.
   (b) Suppose f is continuous everywhere. Prove that $\vec{F}$ is a conservative vector field on $\mathbb{R}^n$.